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Failure of the Alternating Algorithm for Best Approximation of Multivariate Functions*

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The alternating method is an algorithm for obtaining best approximations to elements in a normed space by elements in the vector sum of two subspaces. In its simplest form, the description goes as follows: Let U and V be subspaces of a normed space X. Suppose that there exist proximity maps P: $X \to U$ and Q: $X \to V$. That means that for all x, $\|x - Px\| = \operatorname{dist}(x, U)$ and $\|x - Qx\| = \operatorname{dist}(x, V)$. Starting from any $x_0 \in X$, one computes $x_1 = x_0 - Px_0$, $x_2 = x_1 - Qx_1$, $x_3 = x_2 - Px_2$, and so on. Under favorable circumstances, the sequence $\{x_n\}$ converges to a point z such that $x_0 - z$ is a best approximation to x_0 in $U + V$.

The alternating method apparently originated in 1933 with von Neumann [7]. For a recent survey of the subject see Deutsch's article [2].

In 1951, Diliberto and Strauss [3] showed that the method produces best approximations (in the supremum norm) to a function $x \in C(S \times T)$ by a function of the form $u(s) + v(t)$, with $u \in C(S)$ and $v \in C(T)$. Certain aspects of their work were completed by Aumann in [1]. Part of the article of Golomb [6] deals also with this procedure. A new algorithm of a different type has recently been discovered by von Golitschek [5]. In a recent article

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[4], Dyn has shown that the alternating method fails to produce best approximations of the form

$$x(s, t) \approx u_1(s) + tu_2(s) + v(t)$$

with $x \in C[0, 1]^2$, $u_i \in C[0, 1]$, $v \in C[0, 1]$.

On the other hand, there exist, in a space $C(S \times T)$, subspaces of the form

$$W = C(S) \otimes H + G \otimes C(T)$$

for which the algorithm succeeds. See [9] for the construction of such subspaces, even in cases where G or H are infinite-dimensional. Thus there is some interest in discovering which subspaces of the form W are favorable for the application of the alternating approximation method. We have used the methods of Dyn to prove the following result:

THEOREM. *Let I be a compact interval on the real line. Let G and H be nonzero finite-dimensional Haar subspaces in $C(I)$. If one (or both) of these subspaces has dimension 2 or greater, then the alternating algorithm fails in $C(I^2)$ when applied to the pair of subspaces $G \otimes C(I)$ and $C(I) \otimes H$.*

Proof. Let $n = \dim(G)$ and $m = \dim(H)$. We may assume that $n \geqslant m$ and $n \geqslant 2$. The proof divides into two cases according to whether $m = 1$ or $m > 1$.

The Haar property for G states that no element in G except 0 can vanish at n or more points of I. Equivalently, if $s_1, \ldots, s_n$ are distinct points of I, then the corresponding point functionals $\hat{s}_1, \ldots, \hat{s}_n$ form a basis for the algebraic dual G^*. For $s \in I$ and $x \in C(I)$ we write $\hat{s}(x) = x(s)$. The notation $\Phi \perp G$ signifies that Φ is a continuous linear functional on $C(S)$ and $\Phi(g) = 0$ for all $g \in G$.

Case I, $n \geqslant m \geqslant 2$. Select $s_1 < \cdots < s_{n+3}$ in I and $t_1 < \cdots < t_{m+3}$ in I. Define $f(s_j, t_i) = (-1)^{i+j}$ except for these 9 points, where $f(s_j, t_i) = 0$:

$$(s_{n+2}, t_1),\quad (s_{n+3}, t_1),\quad (s_{n+1}, t_2),\quad (s_{n+2}, t_2),\quad (s_3, t_3),$$
$$(s_1, t_{m+2}),\quad (s_1, t_{m+3}),\quad (s_2, t_{m+2}),\quad (s_2, t_{m+3}).$$

At all other points in $I \times I$ we require only $|f(s, t)| < 1$.

By the theorem of Chebyshev characterizing best approximations, the best approximation to $f(\cdot, t_i)$ in G is 0 for $i = 1, \ldots, m + 3$. Likewise, the best approximation to $f(s_j, \cdot)$ in H is 0 for $j = 1, \ldots, n + 3$. The alternating method is therefore unable to produce an approximation to f better than 0.

The pattern of critical points for f is illustrated for the case $n=4$, $m=3$.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7
t_1	+	−	+	−	+		
t_2	−	+	−	+			−
t_3	+	−		−	+	−	+
t_4	−	+	−	+	−	+	−
t_5			+	−	+	−	+
t_6			−	+	−	+	−

Now it is to be shown that there exists an approximation of f better than 0. This is proved by contradiction. Assume that 0 is a best approximation of f from the subspace $W = G \otimes C(I) + C(I) \otimes H$. By Singer's characterization theorem for best approximations [8, p. 5], there must exist a nonzero linear functional Φ annihilating W, having support in the set of critical points for f, and extremal for f. Since f has only a finite number of critical points, Φ must have the form

$$\Phi = \sum_{i=1}^{m+3} \sum_{j=1}^{n+3} A_{ij} \hat{t}_i \hat{s}_j$$

in which $A_{ij} = 0$ whenever (s_j, t_i) is *not* a critical point of f. Since Φ annihilates W, each "row functional"

$$\sum_{j=1}^{n+3} A_{ij} \hat{s}_j \qquad (1 \leqslant i \leqslant m+3)$$

must annihilate G, as is easily proved. By the Haar property of G, if any row of the matrix A contains 3 zeros, then that row must be 0. By the Haar property of H, if any column of A contains 3 zeros, then that column is 0. By applying this argument repeatedly, we conclude that if either row 1, row 2, column 1, or column 2 contains 3 zeros, then $A = 0$. Hence we assume that in these rows and columns, $A_{ij} = 0$ if and only if (s_j, t_i) is not a critical point of f.

From the Haar property of H, we conclude that the first and second columns of A are proportional. In particular $A_{11} A_{22} = A_{21} A_{12}$. But a consideration of the first two rows of A now will lead to a contradiction. Indeed, the functional

$$\lambda = \sum_{j=1}^{n+3} (A_{21} A_{1j} - A_{11} A_{2j}) \hat{s}_j$$

is nonzero, is supported on n points, and annihilates the n-dimensional Haar space G, which is impossible.

Case II, $n > m = 1$. In this case, select $s_1 < \cdots < s_{n+4}$ in I and $t_1 < \cdots < t_4$ in I. Define $f(s_j, t_i) = (-1)^{i+j}$ except for these special points:

$$f(s_{n+1}, t_1) = f(s_{n+2}, t_1) = f(s_{n+4}, t_1) = 0$$
$$f(s_{n+1}, t_2) = f(s_{n+2}, t_2) = f(s_{n+3}, t_2) = 0$$
$$f(s_{n+4}, t_2) = (-1)^{n+1} = -f(s_{n+4}, t_3)$$
$$f(s_1, t_3) = f(s_2, t_3) = f(s_{n+3}, t_3) = 0$$
$$f(s_1, t_4) = f(s_2, t_4) = f(s_{n+4}, t_4) = 0.$$

At all other points of $I \times I$, $|f(s, t)| < 1$. The rest of the argument is similar to the one given for Case I.

Remarks. The theorem can be generalized so that the domain of f is a set $S \times T$ with $S \subset I$ and $T \subset I$. The proof given above requires that S and T contain certain minimum numbers of points, namely

$$\#T \geqslant 3 + m, \qquad \#S \geqslant 5 + n - \min(m, 2).$$

A separate proof, not given here, shows that the requirement on T can be weakened to

$$\#T \geqslant 4 + m - \min(m, 2).$$

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